Power Series

Radius of Convergence and the Cauchy–Hadamard Formula

Lecture Notes

September 29, 2026

1 Formal Power Series and the Convergence Lemma

Definition (power series)

Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. $\sum_{n=0}^{\infty} a_n z^n$ is a (formal) **power series**. Its n**-th partial sum** is the polynomial $S_n(z) := \sum_{k=0}^{n} a_k z^k$.

Lemma

Suppose that for some $r > 0$ the sequence $(|a_n|r^n)$ is bounded. Then for any z with $|z| < r$, $\sum a_n z^n$ converges. Moreover, if $r' < r$, the series converges uniformly on $\overline{B(0, r')}$.

Remark. The *uniform* convergence in the Lemma need not extend to all of $B(0, r)$, or even to $\overline{B(0, r)}$ itself: consider $\sum z^n$, where $r = 1$. The series converges (even absolutely) at every point of $B(0, 1)$, but not uniformly there; and it diverges at points of $\overline{B(0, 1)}$, e.g. at $z = 1$.

Proof. When $|z| \leq r'$,

$$|a_n z^n| \leq |a_n (r')^n| = |a_n| r^n \Big(\frac{r'}{r}\Big)^n < C\Big(\frac{r'}{r}\Big)^n,$$

using that $|a_n|r^n \leq C$ for some constant C. Since $\sum C(r'/r)^k < \infty$ (a geometric series with ratio < 1), the Weierstrass M-test gives uniform convergence on $\overline{B(0, r')}$. □

Note. If $(|a_n|r^n)$ is *not* bounded, then for z with $|z| = r$, $\sum a_n z^n$ diverges, since some subsequence of $|a_n z^n| = |a_n| r^n$ is unbounded and so does not tend to 0.

Corollary (downward closure)

The set $S := \{t \geq 0 : (|a_n|t^n) \text{ is bounded}\}$ is **downward-closed**: if $s \in S$ and $0 \leq t \leq s$, then $t \in S$. Consequently, every $r' < \sup S$ satisfies $r' \in S$: since r' is not an upper bound for S, some $s \in S$ has $s > r'$, and downward closure gives $r' \in S$.

Proof. If $(|a_n|s^n)$ is bounded and $0 \leq t \leq s$, then $t^n \leq s^n$ for every $n \geq 0$, so $|a_n|t^n \leq |a_n|s^n$; hence $(|a_n|t^n)$ is bounded too. □

2 The Radius of Convergence

Theorem

For any power series $\sum a_n z^n$, let

$$R := \sup\{r : \text{the sequence } (|a_n| r^n) \text{ is bounded}\}.$$

Then:

(1) $\sum a_n z^n$ converges uniformly on $\overline{B(0,r)}$ for any $r < R$;

(2) $\sum a_n z^n$ diverges for any $|z| > R$.

Definition

R above is called the **radius of convergence** of the power series.

Proof. (1) If $r < R$, choose r' with $r < r' < R$; then $r' \in S$ by the Corollary above, so $(|a_n|(r')^n)$ is bounded, and the Lemma gives uniform convergence on $\overline{B(0,r)}$ since $r < r'$.
(2) If $|z| > R$, then $|z| \notin \{t : (|a_n| t^n) \text{ bounded}\}$, so $(|a_n||z|^n)$ is unbounded, and the series diverges by the Note above. □

Cauchy–Hadamard Formula

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} \qquad \Big(\text{convention: } \tfrac{1}{0} = \infty,\ \tfrac{1}{\infty} = 0\Big).$$



Jacques Hadamard
(1865–1963)

Proof. If $r < 1/\limsup_n \sqrt[n]{|a_n|}$, then $r \cdot \limsup_n \sqrt[n]{|a_n|} < 1$, so there exist $a < 1$ and N such that $n > N \implies r\sqrt[n]{|a_n|} < a$, i.e. $r^n|a_n| < a^n < 1$. So $(r^n|a_n|)$ is bounded, hence $r \le R$. This shows $R \ge 1/\limsup_n \sqrt[n]{|a_n|}$.

Conversely, if $(r^n|a_n|)$ is bounded, say $r^n|a_n| \le M$ for all n, then $r\sqrt[n]{|a_n|} \le M^{1/n}$, so

$$\limsup_n r\sqrt[n]{|a_n|} \le \lim_{n\to\infty} M^{1/n} = 1, \qquad \text{so} \qquad r \le \frac{1}{\limsup_n \sqrt[n]{|a_n|}}.$$

So $1/\limsup_n \sqrt[n]{|a_n|}$ is an upper bound for $\{r : (|a_n| r^n) \text{ bounded}\}$, giving $R \le 1/\limsup_n \sqrt[n]{|a_n|}$. □

Summary. $\sum a_n z^n$ has radius of convergence $R = 1/\limsup_n \sqrt[n]{|a_n|}$; it converges uniformly on every disc $\{|z| \le r\}$ with $r < R$, and is continuous on $\{|z| < R\}$.